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# Born renormalization in classical Maxwell electrodynamics

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## Abstract

We define and compute the renormalized four-momentum of the composed physical system: classical Maxwell field interacting with charged point particles. As a ‘reference’ configuration for the field surrounding the particle, we take the Born solution. Unlike in the previous approach [Commun. Math. Phys. 198 (1998) 711; Gen. Relat. Grav. 26 (1994) 167; Acta Phys. Pol. A 85 (1994) 771], based on the Coulomb ‘reference’, a dependence of the four-momentum of the particle (‘dressed’ with the Born solution) upon its acceleration arises in a natural way. This will change the resulting equations of motion. Similarly, we treat the angular momentum tensor of the system. © 2003 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Classical, relativistic electrodynamics is unable to describe interaction between charged particles, intermediated by electromagnetic field. Indeed, typical well posed problems of the theory are of the contradictory nature: either we may solve partial differential equations for the field, with particle trajectories providing sources (given a priori!), or we may solve ordinary differential equations for the trajectories of test particles, with fields providing forces (given a priori!). Combining these two procedures into a single theory leads to

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a contradiction: Lorentz force due to self-interaction is infinite in case of a point particle. Replacing point particle by an extended object is not a good remedy for this disease because it requires a field-theoretical description of the interior of the particle (rigid spheres do not exist in relativity!). This means that the three degrees of freedom of the particle must be replaced by an infinite number of degrees of freedom of the matter fields constituting the particle. Moreover, a highly nonlinear model for the interaction of these fields must be chosen in order to assure the stability of such an object. As a consequence, there is no hope for an effective theory.

There were many attempts to overcome these difficulties. One of them consists in using the Lorentz–Dirac equation, see [2,4,9]. Here, an effective force by which the retarded solution computed for a given particle trajectory acts on that particle is postulated (the remaining field is finite and acts by the usual Lorentz force). Unfortunately, this equation has many drawbacks (cf. Section 8 of [5]). See also [12] for another approach to this problem.

In papers [3,5] a mathematically consistent theory of the physical system “particle(s) + fields” was proposed, which overcomes most of the above difficulties even if some problems still remain. The theory may be defined as follows. We consider a system consisting of charged point particles and the electromagnetic field  $f_{\mu\nu}$ . We always assume that the latter fulfils Maxwell equations with Dirac “delta-like” currents defined uniquely by the particle trajectories. Given such a system, we are able to define its total “renormalized four-momentum”. For a generic choice of fields and particle trajectories this quantity is *not conserved*. Its conservation is an additional condition which we impose on the system. It provides us the missing “equations of motion” for the trajectories and makes the system mathematically closed (cf. [3]).

Definition of the renormalized four-momentum of the system composed of fields and particles, proposed in [5], was based on the following reasoning. Outside of the particles, the contribution to the total four-momentum carried by the Maxwell field  $f_{\mu\nu}$  is given by integrals of the Maxwell energy–momentum tensor-density

$$T^{\mu\nu} = T^{\mu\nu}(f) = \sqrt{-g}(f^{\mu\lambda} f_{\lambda}^{\nu} - \frac{1}{4}g^{\mu\nu} f^{\kappa\lambda} f_{\kappa\lambda}) \quad (1)$$

over a space-like hypersurface  $\Sigma$  (the notation is prepared for working in curvilinear coordinates). Unfortunately, the total integral of this quantity is divergent because of the field singularities at the particle’s positions. The idea proposed in [5] is to consider for each particle a fictitious “reference particle” which moves uniformly along a straight line tangent to the trajectory of the real particle at the point of intersection with  $\Sigma$ . The constant velocity  $\mathbf{u}$  of this hypothetical particle is thus equal to the instantaneous velocity of the real particle at the point of intersection. Give a label ( $i$ ) to each of those hypothetical particles and consider the corresponding Coulomb field  $f_{(i)}^C$  boosted to velocity  $\mathbf{u}_{(i)}$ . In the rest frame of the ( $i$ )th particle, the magnetic and electric components of this field may be written as

$$B_{(i)}^C = 0, \quad (D_{(i)}^C)^k = \frac{e_{(i)}}{4\pi} \frac{x^k}{r^3}. \quad (2)$$

The “reference particle” has the same charge  $e_{(i)}$  and the same rest mass  $m_{(i)}$  as the real particle. By the mass we mean, however, not the “bare mass”, which must later be “dressed” with the energy of its Coulomb tail (which always leads to infinities during renormalization

procedure), but the total energy of the composed system “particle + field” at rest. Hence, the total four-momentum of the  $i$ th reference particle (together with its field  $f_{(i)}^C$ ) equals

$$P_{(i)}^{\nu} = m_{(i)}u_{(i)}^{\nu}. \tag{3}$$

Now, to define the renormalized four-momentum  $p^{C\nu}$  carried by the particles and the field  $f_{\mu\nu}$  surrounding them, we split the energy–momentum density  $T(f)$  into the sum of the reference densities  $T(f_{(i)}^C)$  and the remaining term. According to [5], the remaining term is integrable (more strictly, the principal value of the integral exists), while  $T(f_{(i)}^C)$  terms are already “taken into account” in the four-momenta  $m_{(i)}u_{(i)}$  of the particles (computed at the points of intersection). Hence, the “Coulomb-renormalized four-momentum” of the system is defined by the following formula:

$$p^{C\nu} := P \int_{\Sigma} \left[ T^{\mu\nu}(f) - \sum_i T^{\mu\nu}(f_{(i)}^C) \right] d\sigma_{\mu} + \sum_i m_{(i)}u_{(i)}^{\nu}. \tag{4}$$

It was proved in [5] that  $p^{C\nu}$  depends on  $\Sigma$  only through the points  $A_i$  of intersection of  $\Sigma$  with the trajectories. Next, one postulates that  $p^{C\nu}$  does not depend on those points. This condition implies the dynamics of the particles [5] and makes the evolution of the system unique (cf. [3]).

The above theory is not completely satisfactory, because the subtraction of  $T(f_{(i)}^C)$  in (4) kills only terms which behave like  $r^{-4}$ , while the  $r^{-3}$ -terms remain in (4) and are integrated with  $r^2 dr$  (for simplicity we assume here that  $\Sigma$  near the particle corresponds to  $x^0 = \text{const.}$  in the rest frame). This phenomenon is implied by the analysis of the Maxwell field behaviour in the vicinity of the particle, cf. (5) or Section 5 of [5]. It leads to logarithmic divergencies which disappear only due to the principal value sign  $P$  in front of the integral (4). That sign means that we first compute the integral over  $\Sigma \setminus \mathcal{U}$ , where  $\mathcal{U} = \cup \mathcal{U}_i$  and  $\mathcal{U}_i$  is a small *symmetric* neighbourhood of the  $i$ th particle and then we pass to the limit with  $\mathcal{U}_i$  shrinking to the point:  $\mathcal{U}_i \rightarrow A_i$ . The symmetry is necessary to kill the  $r^{-3}$ -term under integration because it is anti-symmetric.

The main result of the present paper is a new, improved renormalization procedure, which does not rely on the symmetry of  $\mathcal{U}_i$ . We call this new procedure a *Born renormalization*, because the Coulomb reference for a moving particle, matching only its velocity, is here replaced by the Born solution, matching both the velocity and the acceleration of the particle.

We are going to prove in the sequel, that the four-momentum defined via the Coulomb-renormalization is a special case of the result obtained via Born-renormalization, while the physical interpretation of the latter is more natural: all the integrals occurring here are uniquely defined without any use of the principal value sign. Moreover, the ultra-local dependence of the four-momentum upon the acceleration of the particle, implied by the Born renormalization, will change the equations of motion of the particles. That dependence may be also a key to the instability problem of the theory (with an appropriate dependence of the involved functions on the acceleration). We prove in Section 6 that, disregarding this dependence, we recover the previous Coulomb-renormalized formulae.

Our results are based on an analysis of the behaviour of the Maxwell field in the vicinity of the particles done in papers [6,7] (cf. also [2]). Although the asymptotic behaviour of the

radiation field far away from the sources may be found in any textbook, the “near-field” behaviour is less known. The main observation is that—for any choice of particle trajectories—the difference between the retarded and the advanced solution is bounded (in the vicinity of the particles). Hence, we restrict our considerations to the fields which differ from the particle’s retarded (or advanced) field by a term which is bounded in the vicinity of that particle. We also assume that the field at spatial infinity, i.e. for  $r \rightarrow \infty$ , is at most of the order of  $r^{-2}$ . Fields fulfilling those requirements are called *regular*. In the particle’s rest frame, regular fields have the following behaviour near the particle (cf. (25) of [5]):

$$B^k = \tilde{B}^k, \quad D^k = (D^s)^k + \tilde{D}^k, \quad (D^s)^k = \frac{e}{4\pi} \frac{x^k}{r^3} - \frac{e}{8\pi r} \left( a_i \frac{x^i x^k}{r^2} + a^k \right), \quad (5)$$

where  $\tilde{B}$ ,  $\tilde{D}$  are bounded and  $a^k$  are the components of the acceleration of the particle. Above formulae may be proved for the retarded field using Lienard–Wiechert potentials (cf. [2,6,7])<sup>1</sup>. Hence, they are valid for all regular fields.

Our paper is organized as follows. In Sections 2 and 3 we recall and investigate the Fermi-propagated system of coordinates and the Born solution. In Sections 4–7 we restrict ourselves (for simplicity) to the case of a single particle interacting with the field. (A straightforward generalization of these results to the case of many particles is given in Section 8. This generalization does not require any new ingredient because interaction between particles is intermediated via linear Maxwell field.) In Section 4 we define the Born-renormalized four-momentum  $p^{B\nu}$  of the system and prove that it depends on the hypersurface  $\Sigma$  through the point of intersection with the trajectory only. In Section 5 we assume that  $\Sigma$  near the trajectory coincides with the  $x^0 = \text{const.}$  in the Fermi system which allows us to find an explicit expression for  $p^{B\nu}$ . In Section 6 we compare  $p^{B\nu}$  with the Coulomb-renormalized  $p^{C\nu}$  of [5]. The difference of the two is a function of four-velocity and acceleration at the point of intersection. In Section 7 we extend the results of Sections 4–6 to the case of the angular momentum tensor. The fall-off conditions at spatial infinity and technical details of the proofs are presented in Appendices A and B.

We stress that our approach to renormalization never uses any cancellation procedure of the type “ $+\infty - \infty$ ”. Here, everything is finite from the very beginning and the point particle is understood as a mathematical model, approximating a realistic, physical particle which is assumed to be extended. To formulate such a model one has to abandon the idea of a point particle “floating over the field” but rather treat it as a tiny “strong field region” (its internal dynamics is unknown but, probably, highly nonlinear), surrounded by the “week field region”, governed by the linear Maxwell theory. The strong field region (particle’s interior) interacts with the field via its boundary conditions. In other words: the idea to divide “horizontally” the total energy of the system into: (1) the “true material energy” + (2) the free field energy and, finally, (3) the interaction energy which adds to the previous two contributions, must be rejected from the very beginning. Such a splitting, which is

<sup>1</sup> We use this opportunity to correct a missprint in formulae (77)–(79) of [6]: the right-hand sides should be multiplied by  $r$  and the indices below  $B$  on the left-hand sides should be increased by 1. Correct formulae for the arithmetic mean of the retarded and the advanced fields may be found in [7].

possible for linear systems, makes no sense in case of a realistic particle. In our approach, only “vertical” splitting of the energy into contributions contained in disjoint space regions, separated by a chosen boundary, makes sense because of the locality properties of the theory.

The main advantage of the theory constructed this way is its universality: the final result does not depend upon a specific structure of the particle’s interior, which we want to approximate. Moreover (what is even more important!), it does not depend upon a choice of the hypothetical “boundary” which we have used to separate the strong field region from the weak field region: the only assumption is that it is small with respect to characteristic length of the external field.

## 2. The Fermi-propagated system

In this section we recall and investigate the properties of the Fermi-propagated system of coordinates. It is a noninertial system such that the particle is at rest at each instant of time. The use of the Fermi system simplifies considerably description of the field boundary conditions in the vicinity of the particle, given by (2) and (5). The price we pay for this simplification is a bit more complicated (with respect to the inertial system) description of the field dynamics, cf. [5,7].

Let  $y^\lambda$ ,  $\lambda = 0, 1, 2, 3$ , denote the (Minkowski) spacetime coordinates in a fixed inertial (‘laboratory’) system. By  $\mathbf{f}_\lambda = \partial/\partial y^\lambda$  we denote the corresponding orthonormal basis for the metric tensor  $\eta = \text{diag}(-, +, +, +)$ . Let  $q^\lambda(t) = (t, q^k(t))$  be a particle’s trajectory and  $\tau = \tau(t)$  be the particle’s proper time. Then  $d\tau/dt = (1 - v^2)^{1/2}$  where  $v^k = \dot{q}^k$  (dot denotes the derivative w.r.t.  $t$ ). The normalized four-velocity is given by:  $\mathbf{u} = d\mathbf{q}/d\tau = ((1 - v^2)^{-1/2}, (1 - v^2)^{-1/2}v^k)$  and the particle’s acceleration  $\mathbf{a} = d\mathbf{u}/d\tau = d^2q/d\tau^2$ . Clearly,  $(\mathbf{u}|\mathbf{a}) = 0$ .

We define the rest-frame space  $\Sigma_\tau$  as the hyperplane orthogonal to the trajectory (i.e. to  $\mathbf{e}_{(0)} = \mathbf{u}$ ) at  $q(t)$ . Choose any orthonormal basis  $\mathbf{e}_{(l)}$ ,  $l = 1, 2, 3$ , in  $\Sigma_\tau$ , such that  $\mathbf{e}_{(\mu)}$  are positively oriented. Thus  $(\mathbf{e}_{(\alpha)}|\mathbf{e}_{(\beta)}) = \eta_{\alpha\beta}$ . Denote by  $\mathbf{e}_{(l)}(t) = (c_l(t), d_l^k(t))$ ,  $l = 1, 2, 3$  the laboratory components of the triad. We define a new system of coordinates  $x^\mu = (\tau, x^l)$  putting  $y^\lambda = q^\lambda(t) + x^l e_{(l)}^\lambda(t)$ . This is only a local system, defined in a vicinity of the trajectory. For fixed  $\tau$  (or  $t$ ),  $y$  cover the entire  $\Sigma_\tau$  and the particle remains always at the origin  $x^l = 0$ . In coordinates  $(x^\mu)$  the metric tensor equals  $g_{\mu\nu} = (\partial/\partial x^\mu|\partial/\partial x^\nu)$ , where  $\partial/\partial x^\mu \equiv \partial y/\partial x^\mu$ . In particular,  $\partial/\partial \tau = \mathbf{u} + x^l(d\mathbf{e}_{(l)}/d\tau)$ ,  $\partial/\partial x^l = \mathbf{e}_{(l)}$ . Thus  $g_{kl} = \delta_{kl}$ . Orthogonality condition  $(\mathbf{e}_{(l)}|\mathbf{u}) \equiv 0$  implies the following identity:  $(d/d\tau)(\mathbf{e}_{(l)}|\mathbf{u}) = 0$  which means that  $(d\mathbf{e}_{(l)}/d\tau|\mathbf{u}) = -(\mathbf{e}_{(l)}|d\mathbf{u}/d\tau) = -(\mathbf{e}_{(l)}|\mathbf{a}) = -a_l$ , where  $a_l \mathbf{e}_{(l)} = \mathbf{a}$ .

Fermi frame is defined by the following constraint imposed on the triad  $\mathbf{e}_{(l)}$ :  $g_{0l} \equiv N_l = 0$ . This implies that  $d\mathbf{e}_{(l)}/d\tau$  is proportional to  $\mathbf{u}$ ,  $d\mathbf{e}_{(l)}/d\tau = a_l \mathbf{u}$  and determines the propagation of  $\mathbf{e}_{(l)}$  uniquely (provided they are given for  $t = t_0$ ) and consistently (one has  $(d/d\tau)(\mathbf{e}_{(\mu)}|\mathbf{e}_{(\nu)}) = 0$ ). This condition implies  $\dot{c}_l = a_l$ ,  $\dot{d}_l^k = v^k a_l$ . Moreover, one has  $\partial/\partial \tau = N\mathbf{e}_{(0)}$ , where  $N = 1 + a_l x^l$ . Thus  $g_{00} = (N\mathbf{e}_{(0)}|N\mathbf{e}_{(0)}) = -N^2$  (i.e.  $N$  is the lapse function).

In this Fermi-propagated system the field  $f$  is related to the electric and magnetic fields by (cf. (5) and (6) of [7])

$$f^{0k} = N^{-1} D^k, \quad f^{kl} = \epsilon^{klm} B_m.$$

Sometimes it is more convenient to use nonholonomic field coordinates  $f^{(\alpha)(\beta)}$ , calculated w.r.t. the tetrad  $\mathbf{e}_{(\alpha)}$ . They are related to  $f^{\mu\nu}$  by  $f^{\mu\nu} = e^\mu_{(\alpha)} e^\nu_{(\beta)} f^{(\alpha)(\beta)}$  where  $\mu, \nu$  are taken w.r.t.  $(y^\lambda)$  or, alternatively, w.r.t.  $(x^\lambda)$ . In the latter case one has  $e^k_{(l)} = \delta^k_l, e^k_{(0)} = e^0_{(k)} = 0, e^0_{(0)} = N^{-1}$ , which gives  $f^{(0)(k)} = D^k, f^{(k)(l)} = \epsilon^{klm} B_m$ , like in the laboratory frame. Also  $g_{(\alpha)(\beta)} = (\mathbf{e}_{(\alpha)} | \mathbf{e}_{(\beta)}) = \eta_{\alpha\beta}$ . Thus  $T^{(\alpha)(\beta)}$  has the same form as in the laboratory:

$$\begin{aligned} T^{(0)(0)} &= \frac{1}{2}(D^2 + B^2), & T^{(0)(k)} &= T^{(k)(0)} = (D \times B)^k, \\ T^{(k)(l)} &= -D^k D^l - B^k B^l + \frac{1}{2}\delta^{kl}(D^2 + B^2). \end{aligned} \tag{6}$$

We shall use the following

**Proposition 1.** *When integrating over  $\mathcal{O} \subset \Sigma_\tau$ , one can put (in any system of coordinates)*

$$e^\mu_{(\alpha)} d\sigma_\mu = \delta_{\alpha 0} d\Sigma,$$

where  $d\Sigma$  is the volume element for  $\Sigma_\tau$  and  $d\sigma_\mu$  are the basic three-volume forms.

**Proof.** Taking the laboratory frame,  $e^\mu_{(\alpha)} d\sigma_\mu = e^\mu_{(\alpha)} (\partial/\partial y^\mu) \Big] dy^0 \wedge dy^1 \wedge dy^2 \wedge dy^3 = \mathbf{e}_{(\alpha)} \Big] \mathbf{e}^{(0)} \wedge \mathbf{e}^{(1)} \wedge \mathbf{e}^{(2)} \wedge \mathbf{e}^{(3)}$  equals  $\mathbf{e}^{(1)} \wedge \mathbf{e}^{(2)} \wedge \mathbf{e}^{(3)} = d\Sigma$  for  $\alpha = 0$ , but for  $\alpha \neq 0$  it contains  $\mathbf{e}^{(0)}$ , hence it vanishes when we integrate over  $\mathcal{O} \subset \Sigma_\tau$ . □

Now consider the laboratory frame. On each hypersurface  $\Sigma_\tau$  we introduce coordinates  $\tilde{y}^\lambda = y^\lambda - q^\lambda(t) = x^l e^\lambda_{(l)}$  calculated w.r.t. the particle and we decompose the angular momentum tensor-density

$$\mathcal{M}^{\mu\nu\lambda} = y^\nu T^{\mu\lambda} - y^\lambda T^{\mu\nu} \tag{7}$$

as follows:

$$\mathcal{M}^{\mu\nu\lambda} = \tilde{\mathcal{M}}^{\mu\nu\lambda} + q^\nu T^{\mu\lambda} - q^\lambda T^{\mu\nu}, \tag{8}$$

$$\tilde{\mathcal{M}}^{\mu\nu\lambda} = \tilde{y}^\nu T^{\mu\lambda} - \tilde{y}^\lambda T^{\mu\nu}. \tag{9}$$

Here  $\tilde{\mathcal{M}}$  computed at  $y$  is the angular momentum tensor-density w.r.t. the position of the particle  $q(t)$  such that  $y$  belongs to the hyperplane  $\Sigma_\tau$  with  $q(t)$  at its origin.

Integrating over  $\mathcal{O} \subset \Sigma_\tau$  one may use the nonholonomic coordinates  $T^{(\alpha)(\beta)}$  (cf. Proposition 1 and (6)):

$$\begin{aligned} T^{\mu\nu} d\sigma_\mu &= e^\mu_{(\alpha)} e^\nu_{(\beta)} T^{(\alpha)(\beta)} d\sigma_\mu = e^\nu_{(\beta)} T^{(0)(\beta)} d\Sigma, \\ T^{\mu\nu} d\sigma_\mu &= e^\nu_{(0)} \frac{1}{2}(D^2 + B^2) d\Sigma + e^\nu_{(k)} (D \times B)^k d\Sigma, \end{aligned} \tag{10}$$

$$\begin{aligned} \tilde{\mathcal{M}}^{\mu\nu\lambda} d\sigma_\mu &= (x^l e^\nu_{(l)} e^\mu_{(\alpha)} e^\lambda_{(\beta)} T^{(\alpha)(\beta)} - x^l e^\lambda_{(l)} e^\mu_{(\alpha)} e^\nu_{(\beta)} T^{(\alpha)(\beta)}) d\sigma_\mu \\ &= x^l (e^\nu_{(l)} e^\lambda_{(\beta)} - e^\lambda_{(l)} e^\nu_{(\beta)}) T^{(0)(\beta)} d\Sigma, \\ \tilde{\mathcal{M}}^{\mu\nu\lambda} d\sigma_\mu &= (e^\nu_{(l)} e^\lambda_{(0)} - e^\lambda_{(l)} e^\nu_{(0)}) x^l \frac{1}{2}(D^2 + B^2) d\Sigma \\ &\quad + (e^\nu_{(l)} e^\lambda_{(k)} - e^\lambda_{(l)} e^\nu_{(k)}) x^l (D \times B)^k d\Sigma. \end{aligned} \tag{11}$$

### 3. The Born solution

Consider a uniformly accelerated particle, i.e.  $a_l = \text{const}$ . We have  $d\mathbf{u}/d\tau = a_l \mathbf{e}_{(l)}$ ,  $d\mathbf{e}_{(l)}/d\tau = a_l \mathbf{u}$ , which determines its trajectory (a hyperbola) and its Fermi-propagated system uniquely, provided  $q(0) = 0$  and the initial data  $a_l$ ,  $\mathbf{e}_{(l)}$ ,  $\mathbf{u}$  are given. The propagation may be obtained by action of the one-parameter group  $\mathcal{G}$  of proper Lorentz transformations (boosts) on initial data. The group  $\mathcal{G}$  leaves invariant the point  $-e_{(l)}(0)a^l/a^2$ , where  $a = |\mathbf{a}|$ .

We may use a time-independent 3-rotation from  $\mathbf{e}_{(l)}$  to a new triad  $\mathbf{b}_{(l)}$ , such that the acceleration  $\mathbf{a}$  is proportional to the third axis:  $a^l \mathbf{e}_{(l)} \equiv \mathbf{a} = a \mathbf{b}_{(3)}$ . Denote the corresponding Fermi-propagated coordinates by  $x^l$  (for  $\mathbf{e}_{(l)}$ ) and by  $z^l$  (for  $\mathbf{b}_{(l)}$ ). The spherical coordinates related to  $z^l$  are called  $r, \theta, \phi$ . The Born solution of Maxwell equations with a delta-like source carried by the particle (cf. [9–11], Section 3.3 of [8]) reduces in these coordinates to the following time-independent expression:

$$D_r = \frac{e}{\pi r^2} \frac{2 + ar \cos \theta}{(a^2 r^2 + 4 + 4ar \cos \theta)^{3/2}}, \tag{12}$$

$$D_\theta = \frac{e}{\pi r^2} \frac{ar \sin \theta}{(a^2 r^2 + 4 + 4ar \cos \theta)^{3/2}}, \tag{13}$$

$$D_\phi = B_r = B_\theta = B_\phi = 0. \tag{14}$$

The electric field  $D$  is singular not only at  $r = 0$ , where it behaves as in (5) with  $\tilde{D}$  bounded (cf. [8]), but also for  $r = 2/a, \theta = \pi$ . It turns out that the solution describes *two* symmetric particles with opposite charges and opposite accelerations. Actually, the Born solution may be defined as a unique solution of the problem which is invariant with respect to the symmetry group  $\mathcal{G}$  of the problem and satisfies other natural assumptions (cf. [8,11]).

The Fermi propagation consists in acting with the Lorentz rotations (boosts)  $g \in \mathcal{G}$  on the hyperplanes  $\Sigma_\tau$ . This action leaves the 2-plane  $\mathfrak{p} := \{N = 0\} = \{z^3 = -1/a\}$  invariant. The plane splits each  $\Sigma_\tau$  into two half-hyperplanes. Denote by  $P_\tau = \{x \in \Sigma_\tau : z^3 > -1/a\}$  the one which contains our original particle situated at  $r = 0$ .

Assume that  $O \subset P_\tau$  is a small region around the particle described by  $r < R(\theta, \phi)$  where the latter is a given function. In Section 5 we shall need

**Proposition 2.**

$$\begin{aligned} & \int_{P_\tau \setminus O} \frac{1}{2} (D^2 + B^2) d\Sigma \\ &= \int_E \frac{e^2}{2\pi^2 r^2} \frac{\sin \theta dr d\theta d\phi}{(a^2 r^2 + 4 + 4ar \cos \theta)^2} \\ &= \frac{e^2}{2\pi^2} \int_{S^2} \left\{ \frac{1}{16R(\theta, \phi)} + \frac{1}{8} a \cos \theta \log(aR(\theta, \phi)) + O(R) \right\} \sin \theta d\theta d\phi, \end{aligned} \tag{15}$$

where  $E = P_\tau \setminus \mathcal{O} = \{(r, \theta, \phi) : r \geq R(\theta, \phi), r \cos \theta > -1/a\}$  (in spherical coordinates).

$$\begin{aligned} & \int_{P_\tau \setminus \mathcal{O}} z^k \frac{1}{2} (D^2 + B^2) d\Sigma \\ &= \frac{e^2}{32\pi^2} \int_{S^2} \left\{ -\log(aR(\theta, \phi)) \frac{z^k}{r} + O(R) \right\} \sin \theta d\theta d\phi - \frac{e^2}{8\pi} \delta_{3k}. \end{aligned} \tag{16}$$

**Proof.** By lengthy but standard computations. □

The above result can be reformulated using the following construction. For  $G = \sum_k r^k f_k(\theta, \phi)$  we define its singular part  $G_s = \sum_k r^k f_k(\theta, \phi) 1_{[0, d_k]}(r)$  where  $d_k = 0$  for  $k > -3$ ,  $d_k = 1/a$  for  $k = -3$ ,  $d_k = +\infty$  for  $k < -3$  ( $1_B$  is a characteristic function of a set  $B$ , i.e.  $1_B(y) = 0$  for  $y \notin B$ ,  $1_B(y) = 1$  for  $y \in B$ ). Then (cf. (5))

$$\frac{1}{2} [(D^s)^2]_s = \frac{e^2}{32\pi^2} \left( \frac{1}{r^4} - \frac{2a \cos \theta}{r^3} 1_{[0, 1/a]} \right), \tag{17}$$

$$\begin{aligned} & \int_{\Sigma_\tau \setminus \mathcal{O}} \frac{1}{2} [(D^s)^2]_s d\Sigma \\ &= \frac{e^2}{2\pi^2} \int_{S^2} \left\{ \frac{1}{16R(\theta, \phi)} + \frac{1}{8} a \cos \theta \log(aR(\theta, \phi)) \right\} \sin \theta d\theta d\phi, \end{aligned} \tag{18}$$

$$\left[ \frac{1}{2} z^k (D^s)^2 \right]_s = \frac{e^2 z^k}{32\pi^2 r^4} 1_{[0, 1/a]}, \tag{19}$$

$$\int_{\Sigma_\tau \setminus \mathcal{O}} \left[ \frac{1}{2} z^k (D^s)^2 \right]_s d\Sigma = -\frac{e^2}{32\pi^2} \int_{S^2} \log(aR(\theta, \phi)) \frac{z^k}{r} \sin \theta d\theta d\phi. \tag{20}$$

#### 4. The Born-renormalized four-momentum

Throughout the paper we assume that the particle has no internal degrees of freedom, i.e. it is completely characterized by its charge  $e$  and mass  $m$ . Consider a regular Maxwell field  $f$  consistent with the trajectory of the particle (cf. Section 1). We fix a point  $A$  on its trajectory, corresponding to given values of the proper time  $\tau$ , four-velocity  $\mathbf{u}$  and acceleration  $\mathbf{a}$ .

Formula (4) for the Coulomb-renormalized four-momentum was based on the following heuristic picture: A real, physical particle is an extended object, an exact solution of the complete system: “matter fields + electromagnetic field”. The reference particle (passing through  $A$  and moving with the constant four-velocity  $\mathbf{u}$ ) is also an exact, stable solution of the same system, which, moreover, is static (“soliton-like”). Outside of a certain small radius  $r_0$  the matter fields vanish and the electromagnetic field reduces to the Coulomb field  $f^C$ . Hence, for  $\mathcal{U}$  which is very small from the macroscopic point of view but still big from the microscopic point of view (i.e. much bigger than the ball  $K(A, r_0)$  around the particle),



the total amount of the four-momentum carried by the soliton solution and contained in  $\mathcal{U}$  equals:

$$p^{Cv}(\mathcal{U}) = mu^v - \int_{\Sigma_\tau \setminus \mathcal{U}} T^{\mu\nu}(f^C) d\sigma_\mu. \tag{21}$$

The stability assumption means, that for the real particle surrounded by the field  $f$ , the amount of the four-momentum contained in  $\mathcal{U}$  does not differ considerably from the above quantity, provided  $\mathcal{U}$  is very small with respect to the characteristic length of  $f$ . Together with the amount of the four-momentum contained outside of  $\mathcal{U}$

$$p^v(\Sigma_\tau \setminus \mathcal{U}) = \int_{\Sigma_\tau \setminus \mathcal{U}} T^{\mu\nu}(f) d\sigma_\mu, \tag{22}$$

quantity (21) provides, therefore, a good approximation of the total four-momentum of the “extended particle + electromagnetic field” system:

$$p^v \simeq \int_{\Sigma_\tau \setminus \mathcal{U}} [T^{\mu\nu}(f) d\sigma_\mu - T^{\mu\nu}(f^C)] + mu^v. \tag{23}$$

Treating the point particle as an idealization of the extended particle model and applying the above idea, we may shrink  $\mathcal{U}$  to a point, i.e.  $\mathcal{U} \rightarrow A$ , with respect to the macroscopic scale (but keeping  $\mathcal{U}$  always very big with respect to the microscopic scale  $r_0$ ). This procedure—in case of many particles—gives us precisely formula (4).

Now, we assume that also the Born solution has its “extended-particle version”. More precisely, we assume that the total system: “matter fields + electromagnetic field”, admits a stable, stationary (with respect to the one-parameter group  $\mathcal{G}$  of boosts) solution, which coincides with the Born field  $f^B$  outside of a certain small radius  $r_0$  around the particles. This solution represents a pair of uniformly accelerated particles. Denote by  $P^v(\mathbf{u}, \mathbf{a})$  the amount of the total four-momentum carried by this solution in the half-hyperplane  $P_\tau$ . Hence, the amount of the four-momentum contained in  $\mathcal{U}$  equals:

$$p^{Bv}(\mathcal{U}) = P^v(\mathbf{u}, \mathbf{a}) - \int_{P_\tau \setminus \mathcal{U}} T^{\mu\nu}(f^B) d\sigma_\mu. \tag{24}$$

Replacing (21) by (24) in formula (23), we obtain the following approximation for the total four-momentum:

$$\begin{aligned} p^v &\simeq \int_{\Sigma_\tau \setminus \mathcal{U}} T^{\mu\nu}(f) d\sigma_\mu - \int_{P_\tau \setminus \mathcal{U}} T^{\mu\nu}(f^B) d\sigma_\mu + P^v(\mathbf{u}, \mathbf{a}) \\ &= \int_{\Sigma_\tau \setminus \mathcal{O}} T^{\mu\nu}(f) d\sigma_\mu - \int_{P_\tau \setminus \mathcal{O}} T^{\mu\nu}(f^B) d\sigma_\mu \\ &\quad + \int_{\mathcal{O} \setminus \mathcal{U}} [T^{\mu\nu}(f) - T^{\mu\nu}(f^B)] d\sigma_\mu + P^v(\mathbf{u}, \mathbf{a}), \end{aligned} \tag{25}$$

where  $\mathcal{O}$  is a fixed macroscopic neighbourhood of the particle, contained in  $P_\tau$  and containing  $\mathcal{U}$ . Again, treating point particle as an idealization of the extended particle model and applying the above idea, we may pass to the limit  $\mathcal{U} \rightarrow A$  with respect to the macroscopic scale (but keeping  $\mathcal{U}$  always very big with respect to the microscopic scale  $r_0$ ). Unlike in the

Coulomb renormalization, the limit exists without any symmetry assumption about  $\mathcal{U}$ , because  $T(f) - T(f^B)$  behaves like  $r^{-2}$  in the vicinity of the particle (due to formulae (5) and (6) and Section 3). Hence, we obtain the following.

**Definition.** The renormalized four-momentum of the “point particle + electromagnetic field” system is given (in the laboratory system) by

$$\begin{aligned}
 p^{B\nu} := & \int_{\Sigma_\tau \setminus \mathcal{O}} T^{\mu\nu}(f) \, d\sigma_\mu - \int_{P_\tau \setminus \mathcal{O}} T^{\mu\nu}(f^B) \, d\sigma_\mu \\
 & + \int_{\mathcal{O}} [T^{\mu\nu}(f) - T^{\mu\nu}(f^B)] \, d\sigma_\mu + P^\nu(\mathbf{u}, \mathbf{a}).
 \end{aligned}
 \tag{27}$$

The Born field  $f^B$  above is computed assuming that the proper time  $\tau$ ,  $\mathbf{u}$ ,  $\mathbf{a}$  and  $\mathbf{e}_{(l)}$  at  $A$  for both particles (real and uniformly accelerated) coincide. Thus they have the same hyperplane  $\Sigma_\tau$  passing through  $A$  and the same Fermi coordinates  $x^l$  on it.

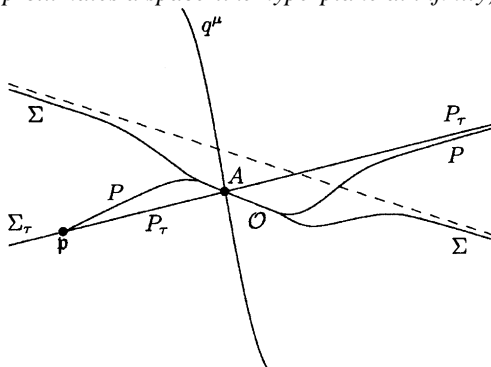
The right-hand side of (27) does not depend, obviously, upon a choice of  $\mathcal{O} \subset P_\tau$ . On the grounds of symmetry we must have  $P^\nu(\mathbf{u}, \mathbf{a}) = m(a)u^\nu + p(a)a^\nu$ , where  $m(a)$  and  $p(a)$  are phenomenological functions of one variable  $a = |\mathbf{a}|$ . We call (27) the *Born-renormalized* four-momentum of the system “point particle + Maxwell field”.

Unfortunately, the above definition cannot be directly generalized to the case of many particle system because, in general, there is no common rest-frame space  $\Sigma_\tau$  for different particles. In what follows we shall rewrite the above definition in a way, which admits an obvious generalization to the case of many particles. For this purpose we replace  $\Sigma_\tau$  by an arbitrary space-like hypersurface  $\Sigma$  which is flat at infinity. More precisely, one has

**Proposition 3.** *Quantity (27) may be rewritten as follows:*

$$\begin{aligned}
 p^{B\nu} = & \int_{\Sigma \setminus \mathcal{O}} T^{\mu\nu}(f) \, d\sigma_\mu + \int_{\mathcal{O}} [T^{\mu\nu}(f) - T^{\mu\nu}(f^B)] \, d\sigma_\mu \\
 & - \int_{P \setminus \mathcal{O}} T^{\mu\nu}(f^B) \, d\sigma_\mu + P^\nu(\mathbf{u}, \mathbf{a}),
 \end{aligned}
 \tag{28}$$

where  $\Sigma, P$  are any space-like hypersurfaces which coincide along some region  $\mathcal{O}$  around  $A$  (we assume that  $P$  has boundary equal to  $\mathfrak{p} = \{z^3 = -1/a\}$ ,  $P$  approximates  $P_\tau$  at infinity and that  $\Sigma$  approximates a space-like hyperplane at infinity) (cf. the figure below).



**Definition.** Hypersurface  $\Sigma$  as in Proposition 3 is called special if  $\Sigma$  coincides with  $\Sigma_\tau$  in a neighbourhood of  $A$ , i.e. if one can take  $P = P_\tau$  (cf. the Noether theorem  $\partial_\mu T^{\mu\nu} = 0$ ).

**Idea of Proof.** First let  $\Sigma$  be special ( $P = P_\tau$ ) and choose  $\mathcal{O}$  contained in  $P_\tau \cap \Sigma$ . Then the first terms in (27) and (28) coincide ( $f$  is a solution of Maxwell equations, we use Noether theorem) and (28) holds. We can assume that in Fermi coordinates  $\mathcal{O} = K(A, R)$ , a ball with a small radius  $R$ . Next we can take any  $\tilde{\Sigma}, \tilde{P}, \tilde{\mathcal{O}}$  as in Proposition 3 and denote the corresponding right-hand side of (28) by  $\tilde{p}^{B\nu}$ . We need to prove  $p^{B\nu} = \tilde{p}^{B\nu}$ . Now we modify the interior of  $\mathcal{O}$ , thus replacing  $\mathcal{O}$  by  $\tilde{\mathcal{O}}'$  without changing its boundary, in such a way that small pieces of  $\tilde{\mathcal{O}}'$  and  $\tilde{\mathcal{O}}$  around  $A$  coincide. It modifies  $p^{B\nu}$  by

$$\left| \left( \int_{\mathcal{O}} - \int_{\tilde{\mathcal{O}}'} \right) [T^{\mu\nu}(f) - T^{\mu\nu}(f^B)] d\sigma_\mu \right| \leq 2 \int_0^R Cr^{-2}r^2 dr = 2CR$$

(cf. Appendix B,  $C = \text{const.}$ ). Next we replace  $\tilde{\mathcal{O}}'$  by its small piece contained in  $\tilde{\mathcal{O}}$ . Finally, we modify  $\Sigma$  and  $P$  outside of that small piece getting  $\tilde{\Sigma}$  and  $\tilde{P}$ , which does not change  $\tilde{p}^{B\nu}$  because  $f, f^B$  are solutions of the Maxwell equations (cf. the Noether theorem and the assumption before (5)). Thus  $p^{B\nu}$  for  $\Sigma$  and  $\tilde{p}^{B\nu}$  for  $\tilde{\Sigma}$  differ by a term of order  $R$ . Taking the limit  $R \rightarrow 0$ , we get  $\tilde{p}^{B\nu} = p^{B\nu}$ .

### 5. Explicit formula for the four-momentum

Here we specify the hypersurface  $\Sigma$  in (28) to be special (i.e.  $A \in \mathcal{O} \subset \Sigma \cap \Sigma_\tau$ ),  $P = P_\tau$  (cf. Section 4) and choose the spherical coordinates related to a Fermi-propagated system as in Section 3.

Let  $\mathcal{U} \subset \mathcal{O}$  be given by  $r < R(\theta, \phi)$ . According to (10), (14) and (15),

$$\begin{aligned} & \int_{P_\tau \setminus \mathcal{U}} T^{\mu\nu}(f^B) d\sigma_\mu \\ &= e^{v_{(0)}} \frac{e^2}{2\pi^2} \int_{S^2} \left\{ \frac{1}{16R(\theta, \phi)} + \frac{1}{8}a \cos \theta \log(aR(\theta, \phi)) + O(R) \right\} \sin \theta d\theta d\phi. \end{aligned}$$

Using (10), (18) and (27), the Born-renormalized four-momentum

$$p^{B\nu} = \int_{\Sigma \setminus \mathcal{O}} T^{\mu\nu}(f) d\sigma_\mu + [e^v_{(\lambda)} K^{(\lambda)}(\mathcal{O}) + P^\nu(\mathbf{u}, \mathbf{a})], \tag{29}$$

where

$$\begin{aligned} K^{(0)}(\mathcal{O}) &= \lim_{\mathcal{U} \rightarrow 0} \left[ \int_{\mathcal{O} \setminus \mathcal{U}} \frac{1}{2} (D^2 + B^2) d\Sigma \right. \\ &\quad \left. - \frac{e^2}{2\pi^2} \int_{S^2} \left\{ \frac{1}{16R(\theta, \phi)} + \frac{1}{8}a \cos \theta \log(aR(\theta, \phi)) \right\} \sin \theta d\theta d\phi \right] \\ &= \int_{\mathcal{O}} \left\{ \frac{1}{2} (D^2 + B^2) - \frac{1}{2} [(D^s)^2]_s \right\} d\Sigma - \int_{\Sigma_\tau \setminus \mathcal{O}} \frac{1}{2} [(D^s)^2]_s d\Sigma, \end{aligned} \tag{30}$$

$$K^{(k)}(\mathcal{O}) = \int_{\mathcal{O}} (D \times B)^k d\Sigma. \tag{31}$$

Now  $\mathcal{O}$  does not need to be inside  $P_\tau$ —only inside  $\Sigma_\tau$ —use (10), (30) and (31).

**6. Relation with the Coulomb-renormalization**

According to (4) and (28), the difference between Born- and Coulomb-renormalized four-momentum

$$p^{Bv} - p^{Cv} = -mu^v + L^v,$$

where

$$L^v = P^v(\mathbf{u}, \mathbf{a}) + \int_{\Sigma \setminus \mathcal{O}} T^{\mu\nu}(f^C) d\sigma_\mu + P \int_{\mathcal{O}} [T^{\mu\nu}(f^C) - T^{\mu\nu}(f^B)] d\sigma_\mu - \int_{P \setminus \mathcal{O}} T^{\mu\nu}(f^B) d\sigma_\mu,$$

which looks like (28) but with the  $P$  sign. Repeating the arguments of Section 5, we get for  $L^v$  an analogue of (29)–(31), again with the  $P$  sign and with  $D, B$  replaced by  $D^C, B^C$ . Setting  $\mathcal{O} = \Sigma_\tau = \Sigma$  and using (2) and (17), one obtains

$$L^v - P^v(\mathbf{u}, \mathbf{a}) = \lim_{R \rightarrow 0} \int_{\Sigma_\tau \setminus K(R)} \left\{ \frac{1}{2}(D^C)^2 - \frac{1}{2}[(D^s)^2]_s \right\} d\Sigma = 0.$$

Thus one gets

**Proposition 4.** *Coulomb- and Born-renormalization of four-momentum give always the same result iff  $P^v(\mathbf{u}, \mathbf{a}) \equiv mu^v$ .*

**7. Born-renormalization of the angular momentum tensor**

In analogy with (27) we define the Born-renormalized tensor of angular momentum

$$M^{Bv\lambda} := \int_{\Sigma \setminus \mathcal{O}} \mathcal{M}^{\mu\nu\lambda}(f) d\sigma_\mu + \int_{\mathcal{O}} [\mathcal{M}^{\mu\nu\lambda}(f) - \mathcal{M}^{\mu\nu\lambda}(f^B)] d\sigma_\mu - \int_{P \setminus \mathcal{O}} \mathcal{M}^{\mu\nu\lambda}(f^B) d\sigma_\mu + M^{v\lambda}(\mathbf{u}, \mathbf{a}) + \frac{e^2}{8\pi a}(u^v a^\lambda - u^\lambda a^v), \tag{32}$$

where  $\mathcal{M}$  was defined in (7),  $\Sigma = \Sigma_\tau, P = P_\tau$ .

The above formula renormalizes the field infinity near the particle, leaving opened the standard convergence problems at spatial infinity ( $r \rightarrow \infty$ ). We discuss briefly these issues in Appendix A. Here, we only mention that these global problems never arise, when the particle’s equations of motion are derived from the momentum and the angular momentum

conservation. Indeed, the conservation condition may always be verified *locally*, i.e. on a family of hypersurfaces  $\tilde{\Sigma}_\tau$  which coincide outside of a certain (spatially compact) world tube  $T$ . Comparing the value of angular momentum calculated on two different  $\tilde{\Sigma}_\tau$  never requires integration outside of  $T$ , because the far-away contributions are the same in both cases.

The last term in (32) could be incorporated into  $M^{\nu\lambda}(\mathbf{u}, \mathbf{a})$  but for the future convenience (see remark at the end of this section) it was written separately. The sum of those two terms can be interpreted as the total angular-momentum of the particle dressed with the Born field. On the symmetry grounds  $\mathbf{M}(\mathbf{u}, \mathbf{a}) = (\mathbf{u} \wedge \mathbf{a})R(a) + (\mathbf{u} \wedge \mathbf{a})^*S(a)$  (one has  $(\mathbf{u} \wedge \mathbf{a})^{\nu\lambda} = u^\nu a^\lambda - u^\lambda a^\nu$ ,  $(\mathbf{u} \wedge \mathbf{a})^* = (\mathbf{e}_{(0)} \wedge \mathbf{a}\mathbf{b}_{(3)})^* = \mathbf{a}\mathbf{b}_{(1)} \wedge \mathbf{b}_{(2)}$ , cf. Section 3). Clearly (32) does not depend on the choice of  $\mathcal{O} \subset P$ . Using Appendices A and B and the Noether theorem  $\partial_\mu \mathcal{M}^{\mu\nu\lambda} = 0$ , one proves (32) for general  $\Sigma$ ,  $P$  as in Proposition 3.

If we restrict ourselves to special hypersurfaces, then using (11) and (16) and relation between  $x^k$  and  $z^k$  on  $\Sigma_\tau$  (Section 3), we get

$$\int_{P \setminus \mathcal{U}} \tilde{\mathcal{M}}^{\mu\nu\lambda}(f^B) d\sigma_\mu = (e_{(l)}^\nu e_{(0)}^\lambda - e_{(l)}^\lambda e_{(0)}^\nu) \times \left[ \frac{e^2}{32\pi^2} \int_{S^2} \left\{ -\log(aR(\theta, \phi)) \frac{x^l}{r} + O(R) \right\} \times \sin \theta d\theta d\phi - \frac{e^2 a^l}{8\pi a} \right],$$

where  $\mathcal{U}$  is given by  $r < R(\theta, \phi)$ . Next, (11), (20) and (32) give (uncontinuous terms of  $a^\lambda/a$  type cancel out!)

$$M^{B\nu\lambda} = \int_{\Sigma \setminus \mathcal{O}} \mathcal{M}^{\mu\nu\lambda}(f) d\sigma_\mu + (e_{(l)}^\nu e_{(\rho)}^\lambda - e_{(l)}^\lambda e_{(\rho)}^\nu) L^{(l)(\rho)}(\mathcal{O}) + [q^\nu(\tau) e_{(\rho)}^\lambda - q^\lambda(\tau) e_{(\rho)}^\nu] K^{(\rho)}(\mathcal{O}) + M^{\nu\lambda}(\mathbf{u}, \mathbf{a}), \tag{33}$$

where

$$L^{(l)(0)}(\mathcal{O}) = \lim_{\mathcal{U} \rightarrow 0} \left[ \int_{\mathcal{O} \setminus \mathcal{U}} \frac{1}{2} x^l (D^2 + B^2) d\Sigma + \frac{e^2}{32\pi^2} \int_{S^2} \log(aR(\theta, \phi)) \frac{x^l}{r} \sin \theta d\theta d\phi \right] = \int_{\mathcal{O}} \left\{ \frac{1}{2} x^l (D^2 + B^2) - \left[ \frac{1}{2} x^l (D^s)^2 \right]_s \right\} d\Sigma - \int_{\Sigma_\tau \setminus \mathcal{O}} \left[ \frac{1}{2} x^l (D^s)^2 \right]_s d\Sigma, \tag{34}$$

$$L^{(l)(k)}(\mathcal{O}) = \int_{\mathcal{O}} x^l (D \times B)^k d\Sigma. \tag{35}$$

Again  $\mathcal{O}$  does not need to be inside  $P$ .

Finally, comparing on  $\Sigma_\tau$  the Born- and Coulomb-renormalization, i.e.

$$M^{C\nu\lambda} = P \int_{\Sigma_\tau} [\mathcal{M}^{\mu\nu\lambda}(f) - \mathcal{M}^{\mu\nu\lambda}(f^C)] d\sigma_\mu$$

(cf. Appendix A), we get that  $M^{B\nu\lambda} - M^{C\nu\lambda}$  equals (32) for  $f = f^C$  and the  $P$  sign before the second term. Thus it equals (33) with  $D, B$  in (30), (31), (34) and (35) replaced by  $D^C, B^C$  and the  $P$  sign in (30). Setting  $\mathcal{O} = \Sigma_\tau = \Sigma$  and using (2) and (19),  $M^{B\nu\lambda} - M^{C\nu\lambda} = M^{\nu\lambda}(\mathbf{u}, \mathbf{a})$ . Therefore, both renormalizations give the same result iff  $M^{\nu\lambda}(\mathbf{u}, \mathbf{a}) = 0$ . This equation was the reason to separate the last term in definition (32) from  $M^{\nu\lambda}(\mathbf{u}, \mathbf{a})$ .

### 8. The case of many particles

Here we extend the results of Sections 4–7 to the case of many particles. For the  $i$ th particle we fix a point  $A_i$  on its trajectory, corresponding to a proper time  $\tau_i$ . The space-like hypersurface  $\Sigma$  passes through all  $A_i$ . At the beginning we assume that some regions  $\mathcal{O}_i \subset \Sigma$  around  $A_i$  are contained in  $P_{\tau_i} \subset \Sigma_{\tau_i}$  corresponding to the  $i$ th particle and that, asymptotically,  $\Sigma$  approximates a space-like hyperplane (special hypersurface). The formula for the Born-renormalized four-momentum generalizes to

$$\begin{aligned}
 p^{Bv} &= \int_{\Sigma \setminus \cup \mathcal{O}_i} T^{\mu\nu}(f) d\sigma_\mu + \sum_i \int_{\mathcal{O}_i} [T^{\mu\nu}(f) - T^{\mu\nu}(f_{(i)}^B)] d\sigma_\mu \\
 &\quad - \sum_i \int_{P_{(i)} \setminus \mathcal{O}_i} T^{\mu\nu}(f_{(i)}^B) d\sigma_\mu + \sum_i P_{(i)}^v(\mathbf{u}_{(i)}, \mathbf{a}_{(i)}),
 \end{aligned}
 \tag{36}$$

$P_{(i)} = P_{\tau_i}$  for the  $i$ th particle. One proves the analogue of Proposition 3. For the special hypersurface (29) generalizes to

$$p^{Bv} = \int_{\Sigma \setminus \cup \mathcal{O}_i} T^{\mu\nu}(f) d\sigma_\mu + \sum_i [e_{(\lambda)}^v K^{(\lambda)}(\mathcal{O}_i) + P_{(i)}^v(\mathbf{u}_{(i)}, \mathbf{a}_{(i)})],
 \tag{37}$$

where  $K^{(\lambda)}(\mathcal{O}_i)$  are given as in (30) and (31) with  $\mathcal{U}_i$  described by  $r < R_{(i)}(\theta, \phi)$ .

The comparison with the Coulomb renormalization gives

$$p^{Bv} - p^{Cv} = \sum_i \{-m_{(i)} u_{(i)}^v + L_{(i)}^v\},$$

where  $L_{(i)}^v$  are as in Section 6 and the analogue of Proposition 4 holds.

The Born-renormalized tensor of angular momentum takes now the form

$$\begin{aligned}
 M^{B\nu\lambda} &= \int_{\Sigma \setminus \cup \mathcal{O}_i} \mathcal{M}^{\mu\nu\lambda}(f) d\sigma_\mu + \sum_i \int_{\mathcal{O}_i} [\mathcal{M}^{\mu\nu\lambda}(f) - \mathcal{M}^{\mu\nu\lambda}(f_{(i)}^B)] d\sigma_\mu \\
 &\quad - \sum_i \int_{P_{(i)} \setminus \mathcal{O}_i} \mathcal{M}^{\mu\nu\lambda}(f_{(i)}^B) d\sigma_\mu \\
 &\quad + \sum_i \left[ M_{(i)}^{\nu\lambda}(\mathbf{u}_{(i)}, \mathbf{a}_{(i)}) + \frac{e_{(i)}^2}{8\pi a_{(i)}} (\mathbf{u}_{(i)} \wedge \mathbf{a}_{(i)})^{\nu\lambda} \right].
 \end{aligned}
 \tag{38}$$

The analogue of Proposition 3 holds. What concerns (33),  $\Sigma \setminus \mathcal{O}$  is now replaced by  $\Sigma \setminus \cup \mathcal{O}_i$  and one has  $\sum_i$  before the remaining  $i$ -dependent terms. For a special  $\Sigma$  one defines the

Coulomb renormalization

$$M^{C\nu\lambda} = P \int_{\Sigma} [\mathcal{M}^{\mu\nu\lambda}(f) - \sum_i \mathcal{M}^{\mu\nu\lambda}(f_{(i)}^C)] d\sigma_{\mu} \tag{39}$$

and shows that the both renormalizations give the same result iff  $M_{(i)}^{\nu\lambda}(\mathbf{u}_{(i)}, \mathbf{a}_{(i)}) \equiv 0$  for all  $i$ .

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### Appendix A. Field fall-off conditions at spatial infinity and a possibility to define global angular momentum

To define the four-momentum of the system, we assume that the field behaves at spatial infinity (i.e. for  $r \rightarrow \infty$ ) like  $r^{-2}$ . To define globally the angular momentum of the system, much stronger fall-off conditions are necessary. Here, we present a possible choice: we assume that the field behaves at spatial infinity like a superposition of boosted Coulomb fields (modulo  $r^{-3}$ -terms). Then (for any space-like hyperplane) the angular momentum density behaves like an anti-symmetric  $r^{-3}$ -term (modulo  $r^{-4}$ -terms). This is sufficient to define global value of angular momentum using the “principal value” sign for integration at infinity. This means that we first integrate over spatially symmetric regions  $V \subset \Sigma$  of an asymptotically flat hypersurface  $\Sigma$  and then pass to the limit  $V \rightarrow \Sigma$ . The symmetry depends upon a choice of a central point  $x_0$ , but it is easy to check that the final result of such a procedure *does not* depend upon this choice. Moreover, the above asymptotic conditions allow us to change  $\Sigma$  at infinity. Indeed, the difference between results obtained for different  $\Sigma$ 's equals to a surface integral at infinity which vanishes as a consequence of the assumed asymptotic conditions (cf. also [1]). We stress, however, that the renormalization proposed in the present paper cures the local and not global problems. Derivation of particle's equations of motion from field equations does not rely on the global problems.

### Appendix B. Approximation by the Born field near the trajectory

Suppose (cf. Section 4) that we have two trajectories: of a real particle  $p$  and of the reference particle  $\tilde{p}$ , which is uniformly accelerated. Both trajectories touch at  $A$ , where the proper times  $\tau_0$ , the four-velocities  $\mathbf{u}$ , the accelerations  $\mathbf{a}$  and  $\mathbf{e}_{(l)}$  coincide. In general, the quantities related to  $\tilde{p}$  differ from those related to  $p$  and are distinguished by tilde. Then  $A \in \Sigma_{\tau_0} = \tilde{\Sigma}_{\tau_0}$ , but for  $H$  approaching  $A$ , one has  $H \in \Sigma_{\tau} \cap \tilde{\Sigma}_{\tilde{\tau}}$  and in general  $\tau \neq \tilde{\tau}$ . Denote by  $r(\tilde{r})$  the radius of  $H$  w.r.t.  $\Sigma_{\tau}$  ( $\tilde{\Sigma}_{\tilde{\tau}}$ ). One has

**Proposition A.1.** *Suppose that  $H$  belongs to a region of space-like directions w.r.t.  $A$ , which is separated from the light cone at  $A$  and that  $H$  approaches  $A$ . Then*

- (1)  $r/\tilde{r}, \tilde{r}/r \sim 1$ ,  
 (2)  $T^{\mu\nu}(f) - T^{\mu\nu}(f^B) \sim r^{-2}$ ,  
 (3)  $\mathcal{M}^{\mu\nu\lambda}(f) - \mathcal{M}^{\mu\nu\lambda}(f^B) \sim r^{-2}$ ,

where  $f$  is a Maxwell field related to  $p$ ,  $f^B$  is the Born solution related to  $\tilde{p}$ .

**Idea of Proof.** We may set  $\tau_0 = 0$ ,  $q(0) = 0$ ,  $u(0) = (1, 0, 0, 0)$ ,  $a(0) = \tilde{a}(0)$ . One has  $\Delta a(\tau) \equiv a(\tau) - \tilde{a}(\tau) \sim \tau$ ,  $\Delta u(\tau) \sim \tau^2$ ,  $\Delta q(\tau) \sim \tau^3$ , the angle between  $\Sigma_\tau$  and  $\tilde{\Sigma}_\tau$  is of order  $\tau^2$ . Denoting by  $\delta F$  the difference of  $F$  computed for  $H$  w.r.t.  $\Sigma_\tau$  and w.r.t.  $\tilde{\Sigma}_\tau$  and using geometric considerations, we get  $\tau \leq Cr$ ,  $\delta\tau \sim r^3$ ,  $\delta x^k \sim r^3$ ,  $\delta r \sim r^3$ ,  $\tilde{r}/r - 1 = \delta r/r \sim r^2$ , (1) follows. Using  $T^{\mu\nu} = e_{(\alpha)}^\mu e_{(\beta)}^\nu T^{(\alpha)(\beta)}$ , (5) and (6), one gets  $T^{\mu\nu} \sim e_{(\alpha)}^\mu e_{(\beta)}^\nu (r^{-2} + ar^{-1} + C)^2 \sim r^{-4}$ ,  $\delta(de_{(\alpha)}^\mu/d\tau) \sim \delta a \sim \tau$ ,  $\delta e_{(\alpha)}^\mu \sim \tau^2 \sim r^2$ ,  $\delta r^{-2} \sim r^{-3} \delta r \sim 1$ ,  $\delta a \sim r$ ,  $\delta r^{-1} \sim r^{-2} \delta r \sim r$ ,  $\delta T \sim r^{-2}$ , (2) holds. Moreover,

$$\begin{aligned} \delta q^\lambda &= \Delta q^\lambda + \tilde{q}^\lambda(\tau) - \tilde{q}^\lambda(\tilde{\tau}) \sim \tau^3 + r^3 \sim r^3, \\ \delta y^\lambda &= \delta q^\lambda + x^l \delta e_{(l)}^\lambda + (\delta x^l) e_{(l)}^\lambda \sim r^3 + r \cdot r^2 + r^3 \cdot 1 \sim r^3, \\ \delta \mathcal{M}^{\mu\nu\lambda} &= y^\nu \delta T^{\mu\lambda} + (\delta y^\nu) T^{\mu\lambda} - (\lambda \leftrightarrow \mu) \sim 1 \cdot r^{-2} + r^3 \cdot r^{-4} \sim r^{-2}, \end{aligned}$$

(3) follows.

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